

Simultaneous Inference of Covariances

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Abstract: We consider asymptotic distributions of maximum deviations of sample covariance matrices, a fundamental problem in high-dimensional inference of covariances. Under mild dependence conditions on the entries of the data matrices, we establish the Gumbel convergence of the maximum deviations. Our result substantially generalizes earlier ones where the entries are assumed to be independent and identically distributed, and it provides a theoretical foundation for high-dimensional simultaneous inference of covariances.

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1. Introduction

Let $\mathbf{X}_n = (X_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$ be a data matrix whose n rows form independent samples from some population distribution with mean vector $\boldsymbol{\mu}_n$ and covariance matrix Σ_n . High dimensional data increasingly occur in modern statistical applications in biology, finance and wireless communication, where the dimension m may be comparable to the number of observations n , or even much larger than n . Therefore, it is necessary to study the asymptotic behavior of statistics of \mathbf{X}_n under the setting that $m = m_n$ grows to infinity as n goes to infinity.

In many empirical examples, it is often assumed that $\Sigma_n = I_m$, where I_m is the $m \times m$ identity matrix, so it is important to perform the test

$$H_0 : \Sigma_n = I_m \quad (1)$$

before carrying out further estimation or inference procedures. Due to high dimensionality, conventional tests often do not work well or cannot be implemented. For example, when $m > n$, the likelihood ratio test (LRT) cannot be used because the sample covariance matrix is singular; and even when $m < n$, the LRT is drifted to infinity and lead to many false rejections if m is also large (Bai et al., 2009). Ledoit and Wolf (2002) found that the empirical distance test

directions (i) reduce the moment condition; (ii) allow a wider range of p ; and (iii) show that some moment condition is necessary. In a recent article, Cai and Jiang (2011) extended those results in two ways: (i) the dimension p could grow exponentially as the sample size n provided exponential moment conditions; and (ii) they showed that the test statistic $\max_{|i-j|>s_n} |\hat{\sigma}_{ij}|$ also converges to the Gumbel distribution if each row of \mathbf{X}_n is Gaussian and is s_n -dependent. The latter generalization is important since it is one of the very few results that allow dependent entries.

In this paper we shall show that a self-normalized version of M_n converges to the Gumbel distribution under mild dependence conditions on the vector (X_{11}, \dots, X_{1m}) . Thus our result provides a theoretical foundation for high-dimensional simultaneous inference of covariances.

The rest of this article is organized as follows. We present the main result in Section 2. In Section 3, we use two examples on linear processes and nonlinear processes to demonstrate that the technical conditions are easily satisfied. We discuss three tests for the covariance structure using our main result in Section 4. The proof is given in Section 5, and some auxiliary results are collected in Section 6.

2. Main result

We consider a slightly more general situation where population distribution can depend on n . Let $\mathbf{X}_n = (X_{n,k,i})_{1 \leq k \leq n, 1 \leq i \leq m}$ be a data matrix whose n rows are i.i.d. m -dimensional random vectors with mean $\boldsymbol{\mu}_n = (\mu_{n,i})_{1 \leq i \leq m}$ and covariance matrix $\Sigma_n = (\sigma_{n,i,j})_{1 \leq i,j \leq m}$. Let x_1, x_2, \dots, x_m be the m columns of \mathbf{X}_n . Let $\bar{x}_i = (1/n) \sum_{k=1}^n X_{n,k,i}$, and write $x_i - \bar{x}_i$ for the vector $x_i - \bar{x}_i \mathbf{1}_n$. The sample covariance between x_i and x_j is defined as

$$\hat{\sigma}_{n,i,j} = \frac{1}{n} (x_i - \bar{x}_i)^\top (x_j - \bar{x}_j).$$

It is unnatural to study the maximum of a collection of random variables which are on different scales, so we consider the normalized version $|\hat{\sigma}_{n,i,j} - \sigma_{n,i,j}|/\tau_{n,i,j}$, where

$$\tau_{n,i,j} = \text{Var}[(X_{n,1,i} - \mu_{n,i})(X_{n,1,j} - \mu_{n,j})].$$

In practice, $\tau_{n,i,j}$ are usually unknown, and can be estimated by

$$\hat{\tau}_{n,i,j} = \frac{1}{n} |(x_i - \bar{x}_i) \circ (x_j - \bar{x}_j) - \hat{\sigma}_{n,i,j} \cdot \mathbf{1}_n|^2.$$

where \circ denotes the Hadamard product defined as $A \circ B := (a_{ij}b_{ij})$ for two matrices $A = (a_{ij})$ and $B = (b_{ij})$ with the same dimensions. We thus consider

$$M_n = \max_{1 \leq i < j \leq m} \frac{|\hat{\sigma}_{n,i,j} - \sigma_{n,i,j}|}{\sqrt{\hat{\tau}_{n,i,j}}}. \quad (2)$$

Due to the normalization procedure, we can assume without loss of generality that $\sigma_{n,i,i} = 1$ and $\mu_{n,i} = 0$ for each $1 \leq i \leq m$.

Define the index set $\mathcal{I}_n = \{(i, j) : 1 \leq i < j \leq m\}$, and for $\alpha = (i, j) \in \mathcal{I}_n$, let $X_{n,\alpha} := X_{n,1,i}X_{n,1,j}$. Define

$$\begin{aligned}\mathcal{K}_n(t, p) &= \sup_{1 \leq i \leq m} \mathbb{E} \exp(t|X_{n,1,i}|^p), \\ \mathcal{M}_n(p) &= \sup_{1 \leq i \leq m} \mathbb{E}(|X_{n,1,i}|^p), \\ \tau_n &= \inf_{1 \leq i < j \leq m} \tau_{n,i,j}, \\ \gamma_n &= \sup_{\alpha, \beta \in \mathcal{I}_n \text{ and } \alpha \neq \beta} |\text{Cor}(X_{n,\alpha}, X_{n,\beta})|, \\ \gamma_n(b) &= \sup_{\alpha \in \mathcal{I}_n} \sup_{\mathcal{A} \subset \mathcal{I}_n, |\mathcal{A}|=b} \inf_{\beta \in \mathcal{A}} |\text{Cor}(X_{n,\alpha}, X_{n,\beta})|.\end{aligned}$$

We need the following technical conditions.

- (A1). $\liminf_{n \rightarrow \infty} \tau_n > 0$.
- (A2). $\limsup_n \gamma_n < 1$.
- (A3). $\gamma_n(b_n) \cdot (\log b_n) = o(1)$ for any sequence (b_n) such that $b_n \rightarrow \infty$.
- (A3'). $\gamma_n(b_n) = o(1)$ for any sequence (b_n) such that $b_n \rightarrow \infty$, and

$$\sum_{\alpha, \beta \in \mathcal{I}_n} [\text{Cov}(X_{n,\alpha}, X_{n,\beta})]^2 = O(m^{4-\epsilon}) \text{ for some constant } \epsilon > 0.$$
- (A4). $\log m = o\left(n^{p/(4+2p)}\right)$ and $\limsup_{n \rightarrow \infty} \mathcal{K}_n(t, p) < \infty$ for some constants $t > 0$ and $0 < p \leq 4$.
- (A4'). $m = O(n^q)$ and $\limsup_{n \rightarrow \infty} \mathcal{M}_n(4q + 4 + \delta) < \infty$ for some constants $q > 0$ and $\delta > 0$.

The two conditions (A3) and (A3') require that the dependence among $X_{n,\alpha}$, $\alpha \in \mathcal{I}_n$, are not too strong. They are translations of (B1) and (B2) in Section 6.1 (see Remark 2 for some equivalent versions), and either of them will make our results valid. We use (A2) to get rid of the case where there may be lots of pairs $(\alpha, \beta) \in \mathcal{I}_n$ such that $X_{n,\alpha}$ and $X_{n,\beta}$ are perfectly correlated. Assumptions (A4) and (A4') connect the growth speed of m relative to n and the moment conditions. They are typical in the context of high dimensional covariance matrix estimation. Condition (A1) excludes the case that $X_{n,\alpha}$ is a constant.

Theorem 2. Suppose that $\mathbf{X}_n = (X_{n,k,i})_{1 \leq k \leq n, 1 \leq i \leq m}$ is a data matrix whose n rows are i.i.d. m -dimensional random vectors, and whose entries have mean zero and variance one. Assume (A1), (A2), either of (A3) and (A3'), and either of (A4) and (A4'), then for any $y \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} P\left(nM_n^2 - 4 \log m + \log(\log m) + \log(8\pi) \leq y\right) = \exp\left(-e^{-y/2}\right).$$

3. Examples

Except for (A4) and (A4'), which put conditions on every single entry of the random vector $(X_{n,1,i})_{1 \leq i \leq m}$, all the other conditions of Theorem 2 are related to the dependence among these entries, which can be arbitrarily complicated. In this section we shall provide examples which satisfy the four conditions (A1), (A2), (A3) and (A3'). Observe that if each row of \mathbf{X}_n is a random vector with uncorrelated entries (specifically, the entries are independent), then all these conditions are automatically satisfied. They are also satisfied if the number of non-zero covariances is bounded.

3.1. Stationary Processes

Suppose $(X_{n,k,i}) = (X_{k,i})$, and each row of $(X_{k,i})_{1 \leq i \leq m}$ is distributed as a stationary process $(X_i)_{1 \leq i \leq m}$ of the form

$$X_i = g(\epsilon_i, \epsilon_{i-1}, \dots)$$

where ϵ_i 's are i.i.d. random variables, and g is a measurable function such that X_i is well-defined. Let $(\epsilon'_i)_{i \in \mathbb{Z}}$ be an i.i.d. copy of $(\epsilon_i)_{i \in \mathbb{Z}}$, and $X'_i = g(\epsilon_i, \dots, \epsilon_1, \epsilon'_0, \epsilon_{-1}, \epsilon_{-2}, \dots)$. Following Wu (2005), define the *physical dependence measure* of order p by

$$\delta_p(i) = \|X_i - X'_i\|_p.$$

Define the squared tail sum

$$\Psi_p(k) = \left[\sum_{j=k}^{\infty} (\delta_p(j))^2 \right]^{1/2},$$

and use Ψ_p as a shorthand for $\Psi_p(0)$.

We give sufficient conditions for (A1), (A2), (A3) and (A3') in the following lemma and leave its proof to the supplementary file.

Lemma 3. (i) If $0 < \Psi_4 < \infty$ and $\text{Var}(X_i X_j) > 0$ for all $i, j \in \mathbb{Z}$, then (A1) holds.
(ii) If in addition, $|\text{Cor}(X_i X_j, X_k X_l)| < 1$ for all i, j, k, l such that they are not all the same, then (A2) holds.
(iii) Assume that the conditions of (i) and (ii) hold. If $\Psi_p(k) = o(1/\log k)$ as $k \rightarrow \infty$, then (A3) holds. If $\sum_{j=0}^m (\Psi_4(j))^2 = O(m^{1-\delta})$ for some $\delta > 0$, then (A3') holds.

Remark 1. Let g be a linear function with $g(\epsilon_i, \epsilon_{i-1}, \dots) = \sum_{j=0}^{\infty} a_j \epsilon_{i-j}$, where ϵ_j are i.i.d. with mean 0 and $\mathbb{E}(|\epsilon_j|^p) < \infty$ and a_j are real coefficients with $\sum_{j=0}^{\infty} a_j^2 < \infty$. Then the physical dependence measure $\delta_p(i) = \|a_i\| \|\epsilon_0 - \epsilon'_0\|_p$. If $a_i = i^{-\beta} \ell(i)$, where $1/2 < \beta < 1$ and ℓ is a slowly varying function, then (X_i)

is a long memory process. Smaller β indicates stronger dependence. Condition (iii) holds for all $\beta \in (1/2, 1)$. Moreover, if $a_i = i^{-1/2}(\log(i))^{-2}$, $i \geq 2$, which corresponds to the extremal case with very strong dependence $\beta = 1/2$, we also have $\Psi_p(k) = O((\log k)^{-3/2}) = o(1/\log k)$. So our dependence conditions are actually quite mild.

If (X_i) is a linear process which is not identically zero, then the following regularity conditions are automatically satisfied: $\Psi_4 > 0$, $\text{Var}(X_i X_j) > 0$ for all $i, j \in \mathbb{Z}$, and $|\text{Cor}(X_i X_j, X_k X_l)| < 1$ for all i, j, k, l such that they are not all the same.

3.2. Non-stationary Linear Processes

Assume that each row of $(X_{n,k,i})$ is distributed as $(X_{n,i})_{1 \leq i \leq m}$, which is of the form

$$X_{n,i} = \sum_{t \in \mathbb{Z}} f_{n,i,t} \epsilon_{i-t},$$

where ϵ_i , $i \in \mathbb{Z}$ are i.i.d. random variables with mean zero, variance one and finite fourth moment, and the sequence $(f_{n,i,t})$ satisfies $\sum_{t \in \mathbb{Z}} f_{n,i,t}^2 = 1$. Denote by κ_4 the fourth cumulant of ϵ_0 . For $1 \leq i, j, k, l \leq m$, we have

$$\sigma_{n,i,j} = \sum_{t \in \mathbb{Z}} f_{n,i,i-t} f_{n,j,j-t},$$

$$\text{Cov}(X_{n,i} X_{n,j}, X_{n,k} X_{n,l}) = \text{Cum}(X_{n,i}, X_{n,j}, X_{n,k}, X_{n,l}) + \sigma_{n,i,k} \sigma_{n,j,l} + \sigma_{n,i,l} \sigma_{n,j,k},$$

where $\text{Cum}(X_{n,i}, X_{n,j}, X_{n,k}, X_{n,l})$ is the fourth order joint cumulant of the random vector $(X_{n,i}, X_{n,j}, X_{n,k}, X_{n,l})^\top$, which can be expressed as

$$\text{Cum}(X_{n,i}, X_{n,j}, X_{n,k}, X_{n,l}) = \sum_{t \in \mathbb{Z}} f_{n,i,i-t} f_{n,j,j-t} f_{n,k,k-t} f_{n,l,l-t} \kappa_4,$$

by the multilinearity of cumulants. In particular, we have

$$\text{Var}(X_i X_j) = 1 + \sigma_{n,i,j}^2 + \kappa_4 \cdot \sum_{t \in \mathbb{Z}} f_{n,i,t}^2 f_{n,j,t}^2.$$

Since $\kappa_4 = \text{Var}(\epsilon_0^2) - 2(\mathbb{E}\epsilon_0^2)^2 \geq -2$, the condition

$$\kappa_4 > -2 \tag{3}$$

guarantees (A1) in view of

$$\text{Var}(X_i X_j) \geq (1 + \sigma_{n,i,j}^2)(1 + \min\{\kappa/2, 0\}) \geq \min\{1, 1 + \kappa/2\} > 0.$$

To ensure the validity of (A2), it is natural to assume that no pairs $X_{n,i}$ and $X_{n,j}$ are strongly correlated, *i.e.*

$$\limsup_{n \rightarrow \infty} \sup_{1 \leq i < j \leq m} \left| \sum_{t \in \mathbb{Z}} f_{n,i,i-t} f_{n,j,j-t} \right| < 1. \tag{4}$$

We need the following lemma, whose proof is elementary and will be given in the supplementary file.

Lemma 4. *The condition (4) suffices for (A2) if ϵ_i 's are i.i.d. $N(0, 1)$.*

As an immediate consequence, when ϵ_i 's are i.i.d. $N(0, 1)$, we have

$$\ell := \limsup_{n \rightarrow \infty} \inf_{*} \inf_{\rho \in \mathbb{R}} \text{Var}(X_{n,i}X_{n,j} - \rho X_{n,k}X_{n,l}) > 0,$$

where \inf_{*} is taken over all $1 \leq i, j, k, l \leq m$ such that $i < j$, $k < l$ and $(i, j) \neq (k, l)$. Observe that when ϵ_i 's are i.i.d. $N(0, 1)$,

$$\begin{aligned} \text{Var}(X_{n,i}X_{n,j} - \rho X_{n,k}X_{n,l}) &= 2 \cdot \sum_{t \in \mathbb{Z}} (f_{n,i,i-t}f_{n,j,j-t} - \rho f_{n,k,k-t}f_{n,l,l-t})^2 \\ &\quad + \sum_{s < t} (f_{n,i,i-t}f_{n,j,j-s} + f_{n,i,i-s}f_{n,j,j-t} \\ &\quad - \rho f_{n,k,k-t}f_{n,l,l-s} - \rho f_{n,k,k-s}f_{n,l,l-t})^2; \end{aligned} \quad (5)$$

and when ϵ_i 's are arbitrary variables, the variance is given by the same formula with the number 2 in (5) being replaced by $2 + \kappa_4$. Therefore, if (3) holds, then

$$\limsup_{n \rightarrow \infty} \inf_{*} \inf_{\rho \in \mathbb{R}} \text{Var}(X_{n,i}X_{n,j} - \rho X_{n,k}X_{n,l}) \geq \min\{1, 1 + \kappa_4/2\} \cdot \ell > 0,$$

which implies (A2) holds. To summarize, we have shown that (3) and (4) suffice for (A2).

Now we turn to Conditions (A3) and (A3'). Set

$$h_n(k) = \sup_{1 \leq i \leq m} \left(\sum_{|t|=\lfloor k/2 \rfloor}^{\infty} f_{n,i,t}^2 \right)^{1/2},$$

where $\lfloor x \rfloor = \max\{y \in \mathbb{Z} : y \leq x\}$ for any $x \in \mathbb{E}$, then we have

$$|\sigma_{n,i,j}| \leq 2h_n(0)h_n(|i-j|) = 2h_n(|i-j|).$$

Fixing a subset $\{i, j\}$, for any integer $b > 0$, there are at most $8b^2$ subsets $\{k, l\}$ such that $\{k, l\} \subset B(i; b) \cup B(j; b)$, where $B(x; r)$ is the open ball $\{y : |x-y| < r\}$. For all other subsets $\{k, l\}$, we have

$$|\text{Cov}(X_{n,i}X_{n,j}, X_{n,k}X_{n,l})| \leq (4 + 2\kappa_4)h_n(b),$$

and hence (A3) holds if we assume $h_n(k_n) \log k_n = o(1)$ for any positive sequence (k_n) such that $k_n \rightarrow \infty$. (A3') holds if we assume

$$\sum_{k=1}^m [h_n(k)]^2 = O(m^{1-\delta}).$$

for some $\delta > 0$, because

$$|\text{Cov}(X_{n,i}X_{n,j}, X_{n,k}X_{n,l})| \leq 2\kappa_4 h_n(|i-j|) + 2h_n(|i-k|) + 2h_n(|i-l|).$$

4. Testing for covariance structures

The asymptotic distribution given in Theorem 2 has several statistical applications. One of them is in high dimensional covariance matrix regularization, because Theorem 2 implies a uniform convergence rate for all sample covariances. Recently, Cai and Liu (2011) explored this direction, and proposed a thresholding procedure for sparse covariance matrix estimation, which is adaptive to the variability of each individual entry. Their method is superior to the uniform thresholding approach studied by Bickel and Levina (2008b).

Testing structures of covariance matrices is also a very important statistical problem. As mentioned in the introduction, when the data dimension is high, conventional tests often cannot be implemented or do not work well. Let Σ_n and R_n be the covariance matrix and correlation matrix of the random vector $(X_{n,1,i})_{1 \leq i \leq m}$ respectively. Two types of tests have been studied under the large n , large m paradigm. Chen et al. (2010), Bai et al. (2009), Ledoit and Wolf (2002) and Johnstone (2001) considered the test

$$H_0 : \Sigma_n = I_m; \quad (6)$$

and Liu et al. (2008), Schott (2005), Srivastava (2005) and Jiang (2004) studied the problem of testing for complete independence

$$H_0 : R_n = I_m. \quad (7)$$

Their testing procedures are all based on the critical assumption that the entries of the data matrix \mathbf{X}_n are i.i.d., while the hypotheses themselves only require the entries of $(X_{n,1,i})_{1 \leq i \leq m}$ to be uncorrelated. Evidently, we can use M_n in (2) to test (7), and we only require the uncorrelatedness for the validity of the limiting distribution established in Theorem 2, as long as the mild conditions of the theorem are satisfied. On the other hand, we can also take the sample variances into consideration, and use the following test statistic

$$M'_n = \max_{1 \leq i \leq j \leq m} \frac{|\hat{\sigma}_{n,i,j} - \sigma_{n,i,j}|}{\sqrt{\hat{\tau}_{n,i,j}}}.$$

to test the identity hypothesis (6), where $\sigma_{n,i,j} = I\{i = j\}$. It is not difficult to verify that M'_n has the same asymptotic distribution as M_n under the same conditions with the only difference being that we now have to take sample variances into account as well, namely, the index set \mathcal{I}_n in Section 2 is redefined as $\mathcal{I}_n = \{(i, j) : 1 \leq i \leq j \leq m\}$. Clearly, we can also use M'_n to test $H_0 : \Sigma_n = \Sigma^0$ for some known covariance matrix Σ^0 .

By checking the proof of Theorem 2, it can be seen that if instead of taking the maximum over the set $\mathcal{I}_n = \{(i, j) : 1 \leq i < j \leq m\}$, we only take the maximum over some subset $A_n \subset \mathcal{I}_n$ whose cardinality $|A_n|$ converges to infinity, then the maximum also has the Gumbel type convergence with normalization constants which are functions of the cardinality of the set A_n . Based on this observation, we are able to consider three more testing problems.

4.1. Test for stationarity

Suppose we want to test whether the population is a stationary time series. Under the null hypothesis, each row of the data matrix \mathbf{X}_n is distributed as a stationary process $(X_i)_{1 \leq i \leq m}$. Let $\gamma_l = \text{Cov}(X_0, X_l)$ be the autocovariance at lag l . In principle, we can use the following test statistic

$$\tilde{T}_n = \max_{1 \leq i \leq j \leq m} \frac{|\hat{\sigma}_{n,i,j} - \gamma_{i-j}|}{\sqrt{\hat{\tau}_{n,i,j}}}.$$

The problem is that γ_l are unknown. Fortunately, they can not only be estimated, but also be estimated with higher accuracy

$$\hat{\gamma}_{n,l} = \frac{1}{nm} \sum_{k=1}^n \sum_{i=|l|+1}^n (X_{n,k,i-|l|} - \hat{\mu}_n)(X_{n,k,i} - \hat{\mu}_n),$$

where $\hat{\mu}_n = (1/nm) \sum_{k=1}^n \sum_{i=1}^m X_{n,k,i}$, and we are lead to the test statistic

$$T_n = \max_{1 \leq i \leq j \leq m} \frac{|\hat{\sigma}_{n,i,j} - \hat{\gamma}_{i-j}|}{\sqrt{\hat{\tau}_{n,i,j}}}.$$

Using similar arguments of Theorem 2 of Xiao and Wu (2011), under suitable conditions, we have

$$\max_{0 \leq l \leq m-1} |\hat{\gamma}_{n,l} - \gamma_l| = O_P(\sqrt{\log m/nm}).$$

Therefore, the limiting distribution for M_n in Theorem 2 also holds for T_n .

4.2. Test for bandedness

In time series and longitudinal data analysis, it can be of interest to test whether Σ_m has the banded structure. The hypothesis to be tested is

$$H_0 : \sigma_{n,i,j} = 0 \text{ if } |i-j| > B, \quad (8)$$

where $B = B_n$ may depend on n . Cai and Jiang (2011) studied this problem under the assumption that each row of the data matrix \mathbf{X}_n is a Gaussian random vector. They proposed to use the maximum sample correlation outside the band

$$\tilde{T}_n = \max_{|i-j| > B} \frac{\hat{\sigma}_{n,i,j}}{\sqrt{\hat{\sigma}_{n,i,i} \hat{\sigma}_{n,j,j}}}$$

as the test statistic, and proved that T_n also has the Gumbel type convergence provided that $B_n = o(m)$ and several other technical conditions hold.

Apparently, our Theorem 2 can be employed to test (8). If all the conditions of the theorem are satisfied, the test statistic

$$T_n = \max_{|i-j| > B_n} \frac{|\hat{\sigma}_{n,i,j}|}{\sqrt{\hat{\tau}_{n,i,j}}}.$$

has the same asymptotic distribution as M_n as long as $B_n = o(m)$. Our theory does not need the normality assumption.

4.3. Assess the tapering procedure

Banding and tapering are commonly used regularization procedures in high dimensional covariance matrix estimation. Convergence rates were first obtained by Bickel and Levina (2008a), and later on improved by Cai et al. (2010). Let us introduce a weaker version of the latter result. Suppose each row of \mathbf{X}_n is distributed as the random vector $X = (X_i)_{1 \leq i \leq m}$ with mean μ and covariance matrix $\Sigma = (\sigma_{ij})$. Let K_0, K and t be positive constants, and $\mathcal{C}_\eta(K_0, K, t)$ be the class of m -dimensional distributions which satisfy the following conditions

$$\begin{aligned} \max_{|i-j|=k} |\sigma_{ij}| &\leq Kk^{-(1+\eta)} \quad \text{for all } k; \\ \lambda_{\max}(\Sigma) &\leq K_0; \\ P[|v^\top(X - \mu)| > x] &\leq e^{-tx^2/2} \quad \text{for all } x > 0 \text{ and } \|v\| = 1; \end{aligned} \quad (9)$$

where $\lambda_{\max}(\Sigma)$ is the largest eigenvalue of Σ . For a given even integer $1 \leq B \leq m$, define the tapered estimate of the covariance matrix Σ

$$\hat{\Sigma}_{n, B_n} = (w_{ij} \hat{\sigma}_{n, i, j}),$$

where the weights correspond to a flat top kernel and are given by

$$w_{ij} = \begin{cases} 1, & \text{when } |i - j| \leq B_n/2, \\ 2 - 2|i - j|/B_n, & \text{when } B_n/2 < |i - j| \leq B_n, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 5 (Cai et al., 2010). *If $m \geq n^{1/(2\eta+1)}$, $\log m = o(n)$ and $B_n = n^{1/(2\eta+1)}$, then there exists a constant $C > 0$ such that*

$$\sup_{\mathcal{C}_\eta} \mathbb{E} \left[\lambda(\hat{\Sigma}_{n, B_n} - \Sigma) \right]^2 \leq Cn^{-2\eta/(2\eta+1)} + C \frac{\log m}{n}.$$

We see that it is the parameter η that decides the convergence rate under the operator norm. After such a tapering procedure has been applied, it is important to ask whether it is appropriate, and in particular, whether (9) is satisfied. We propose to use

$$T_n = \max_{|i-j| > B_n} \frac{|\hat{\sigma}_{n, i, j}|}{\sqrt{\hat{\tau}_{n, i, j}}}$$

as the test statistic. According to the observation made at the beginning of Section 4, if the conditions of Theorem 2 are satisfied, then

$$T'_n = \max_{|i-j| > B_n} \frac{|\hat{\sigma}_{n, i, j} - \sigma_{i, j}|}{\sqrt{\hat{\tau}_{n, i, j}}}$$

has the same limiting law as M_n . On the other hand, (9) implies that

$$\max_{|i-j| > B_n} |\sigma_{i, j}| = O\left(n^{-(1+\eta)/(2\eta+1)}\right),$$

so T_n has the same limiting distribution as T'_n if we further assume $\log m = o(n^{2/(4\eta+2)})$.

5. Proof

The proofs of Theorem 2 under (A4) and (A4') are very similar, and they share a common Poisson approximation step, which we will formulate in Section 5.1 under a more general context, where the limiting distribution of the maximum of sample means is obtained. Since the proof under (A4') is more involved, we provide the detailed proof under this assumption in Section 5.2, and point out in Section 5.3 how it can be adapted to give a proof under (A4).

5.1. Maximum of Sample Means: An Intermediate Step

In this section we provide a general result on the maximum of sample means. Let $\mathbf{Y}_n = (Y_{n,k,i})_{1 \leq k \leq n, i \in \mathcal{I}_n}$ be a data matrix whose n rows are independent and identically distributed, and whose entries have mean zero and variance one, where \mathcal{I}_n is an index set with cardinality $|\mathcal{I}_n| = s_n$. For each $i \in \mathcal{I}_n$, let y_i be the i -th column of \mathbf{Y}_n , $\bar{y}_i = (1/n) \sum_{k=1}^n Y_{n,k,i}$. Define

$$W_n = \max_{i \in \mathcal{I}_n} |\bar{y}_i|. \quad (10)$$

Let Σ_n be the covariance matrix of the s_n -dimensional random vector $(Y_{n,1,i})_{i \in \mathcal{I}_n}$.

Lemma 6. *Assume Σ_n satisfies either (B1) or (B2) of Section 6.1 and $\log s_n = o(n^{1/3})$. Suppose there is a constant $C > 0$ such that $Y_{n,k,i} \in \mathcal{B}(1, Ct_n)$ for each $1 \leq k \leq n$, $i \in \mathcal{I}_n$, with*

$$t_n = \frac{\sqrt{n}\delta_n}{(\log s_n)^{3/2}},$$

where (δ_n) is a sequence of positive numbers such that $\delta_n = o(1)$ and $(\log s_n)^3/n = o(\delta_n)$, and the definition of the collection $\mathcal{B}(d, \tau)$ is given in (27). Then

$$\lim_{n \rightarrow \infty} P(nW_n^2 - 2 \log s_n + \log(\log s_n) + \log \pi \leq z) = \exp(-e^{-z/2}). \quad (11)$$

Proof. For each $z \in \mathbb{R}$, let $z_n = a_{2s_n}z/2 + b_{2s_n}$. Let $(Z_{n,i})_{i \in \mathcal{I}_n}$ be a mean zero normal random vector with covariance matrix Σ_n . For any subset $A = \{i_1, i_2, \dots, i_d\} \subset \mathcal{I}_n$, let $y_A = \sqrt{n}(\bar{y}_{i_1}, \bar{y}_{i_2}, \dots, \bar{y}_{i_d})^\top$ and $Z_A = (Z_{i_1}, Z_{i_2}, \dots, Z_{i_d})$. By Lemma 8, we have for $\theta_n = \delta_n^{1/2}/\sqrt{\log s_n}$ that

$$\begin{aligned} P(|y_A|_\bullet > z_n) &\leq P(|Z_A|_\bullet > z_n - \theta_n) + C_d \exp\left\{-\frac{\theta_n}{C_d \delta_n (\log s_n)^{-3/2}}\right\} \\ &\leq P(|Z_A|_\bullet > z_n - \theta_n) + C_d \exp\left\{-(\log s_n) \delta_n^{-1/2}\right\} \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{A \subset \mathcal{I}_n, |A|=d} P(|y_A|_\bullet > z_n) \\ \leq \sum_{A \subset \mathcal{I}_n, |A|=d} P(|Z_A|_\bullet > z_n - \theta_n) + C_d s_n^d \exp\left\{-(\log s_n) \delta_n^{-1/2}\right\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & \sum_{A \subset \mathcal{I}_n, |A|=d} P(|y_A|_{\bullet} > z_n) \\ & \geq \sum_{A \subset \mathcal{I}_n, |A|=d} P(|Z_A|_{\bullet} > z_n + \theta_n) - C_d s_n^d \exp \left\{ -(\log s_n) \delta_n^{-1/2} \right\}. \end{aligned}$$

Since $(z_n \pm \theta_n)^2 = 2 \log s_n - \log(\log s_n) - \log \pi + z + o(1)$, by Lemma 7, we know

$$\lim_{n \rightarrow \infty} \sum_{A \subset \mathcal{I}_n, |A|=d} P(|Z_A|_{\bullet} > z_n \pm \theta_n) = \frac{e^{-dz/2}}{d!},$$

and hence

$$\lim_{n \rightarrow \infty} \sum_{A \subset \mathcal{I}_n, |A|=d} P(|y_A|_{\bullet} > z_n) = \frac{e^{-dz/2}}{d!}.$$

The proof is complete in view of Lemma 9. \square

5.2. Proof under (A4')

We divide the proof into three steps. The first one is a truncation step, which will make the Gaussian approximation result Lemma 8 and the Bernstein inequality applicable, so that we can prove Theorem 2 under the assumption that all the involved mean and variance parameters are known. In the next two steps we show that plugging in estimated mean and variance parameters does not change the limiting distribution.

Step 1: Truncation For notational simplicity we let $q = p/(4 + 2p)$. Define

$$\tilde{X}_{n,k,i} = X_{n,k,i} I \left\{ |X_{n,k,i}| \leq n^{1/(4+2p)} \right\}, \quad (12)$$

and define \tilde{M}_n similarly as M_n with $X_{n,k,i}$ being replaced by its truncated version $\tilde{X}_{n,k,i}$. Since $\log m = o(n^q)$, we have

$$\begin{aligned} P(\tilde{M}_n \neq M_n) & \leq \sum_{k=1}^n \sum_{i=1}^m P \left[|X_{n,k,i}| > n^{1/(4+2p)} \right] \\ & \leq nm \mathcal{K}_n(t, p) \exp \left\{ -tn^{p/(4+2p)} \right\} \\ & = \mathcal{K}_n(t, p) \exp \left\{ -tn^q + \log m + \log n \right\} = o(1). \end{aligned}$$

Therefore, in the rest of the proof, it suffices to consider $\tilde{X}_{n,k,i}$. For notational simplicity, we still use $\tilde{X}_{n,k,i}$ to denote its centered version with mean zero.

Define $\tilde{\sigma}_{n,i,j} = \mathbb{E}(\tilde{X}_{n,1,i}\tilde{X}_{n,1,j})$, and $\tilde{\tau}_{n,i,j} = \text{Var}(\tilde{X}_{n,1,i}\tilde{X}_{n,1,j})$. Set

$$M_{n,1} = \max_{1 \leq i < j \leq m} \frac{1}{\sqrt{\tilde{\tau}_{n,i,j}}} \left| \frac{1}{n} \sum_{k=1}^n \tilde{X}_{n,k,i} \tilde{X}_{n,k,j} - \tilde{\sigma}_{n,i,j} \right|;$$

$$M_{n,2} = \max_{1 \leq i < j \leq m} \frac{1}{\sqrt{\tilde{\tau}_{n,i,j}}} \left| \frac{1}{n} \sum_{k=1}^n \tilde{X}_{n,k,i} \tilde{X}_{n,k,j} - \sigma_{n,i,j} \right|.$$

Elementary calculations show that

$$\max_{1 \leq i \leq j \leq m} |\tilde{\sigma}_{n,i,j} - \sigma_{n,i,j}| \leq C \exp\{-tn^q/2\}, \quad \text{and} \quad (13)$$

$$\max_{\alpha, \beta \in \mathcal{I}_n} \left| \text{Cov}(\tilde{X}_{n,\alpha}, \tilde{X}_{n,\beta}) - \text{Cov}(X_{n,\alpha}, X_{n,\beta}) \right| \leq C \exp\{-tn^q/2\}. \quad (14)$$

By (14), we know the covariance matrix of $(\tilde{X}_{n,\alpha})_{\alpha \in \mathcal{I}_n}$ satisfies either (B1) or (B2) if Σ_n satisfies (B1) or (B2) correspondingly. On the other hand, we have by elementary calculation that there exists a constant $C_p > 0$ such that

$$\limsup_{n \rightarrow \infty} \max_{\alpha \in \mathcal{I}_n} \mathbb{E} \exp\{C_p t |\tilde{X}_{n,\alpha}|^{p/2}\} < \infty.$$

It follows that when $0 < p < 2$, for each integer $r \geq 3$

$$\begin{aligned} \mathbb{E} |\tilde{X}_{n,\alpha}|^r &\leq \mathbb{E} |\tilde{X}_{n,\alpha}|^{rp/2} \cdot \left(4n^{2/(4+2p)}\right)^{r(1-p/2)} \\ &\leq \left(4n^{2/(4+2p)}\right)^{r(1-p/2)} r! (C_p t)^{-r} \mathbb{E} \exp\{C_p t |X_{n,\alpha}|^{p/2}\}. \end{aligned}$$

Therefore,

$$\mathbb{E}_0 \tilde{X}_{n,\alpha} \in \mathcal{B} \left[1, C \frac{\sqrt{n}}{n^{2p/(4+2p)}} \right].$$

When $2 \leq p \leq 4$, it is easily seen that $\mathbb{E}_0 \tilde{X}_{n,\alpha} \in \mathcal{B}(1, C)$. Since $\log m = o(n^q)$, we know all the conditions of Lemma 6 are satisfied, and hence

$$\lim_{n \rightarrow \infty} P(nM_{n,1}^2 - 4 \log m + \log(\log m) + \log(8\pi) \leq y) = \exp(-e^{-y/2}). \quad (15)$$

Combining (13) and (14), we know the preceding equation (15) also holds with $M_{n,1}$ being replaced by $M_{n,2}$.

Step 2: Effect of Estimated Means Set $\bar{X}_{n,i} = (1/n) \sum_{k=1}^n \tilde{X}_{n,k,i}$. Define

$$M_{n,3} = \max_{1 \leq i < j \leq m} \frac{1}{\sqrt{\tilde{\tau}_{n,i,j}}} \left| \frac{1}{n} \sum_{k=1}^n (\tilde{X}_{n,k,i} - \bar{X}_{n,i})(\tilde{X}_{n,k,j} - \bar{X}_{n,j}) - \sigma_{n,i,j} \right|.$$

In this step we show that (15) also holds for $M_{n,3}$. Observe that

$$|M_{n,3} - M_{n,2}| \leq \max_{1 \leq i < j \leq m} \frac{|\bar{X}_{n,i} \bar{X}_{n,j}|}{\sqrt{\tilde{\tau}_{n,i,j}}} \leq \max_{1 \leq i \leq m} |\bar{X}_{n,i}|^2 \cdot \left(\min_{1 \leq i < j \leq m} \tilde{\tau}_{n,i,j} \right)^{-1/2}.$$

Since each $X_{n,k,i}$ is bounded by $2n^{1/(4+2p)}$, by Bernstein's inequality we have for any constant $K > 0$,

$$\begin{aligned} \max_{1 \leq i \leq m} P \left(|\bar{X}_{n,i}| > 2K \sqrt{\frac{\log m}{n}} \right) &\leq C \exp \left\{ -\frac{2K^2 n \log m}{Cn + 2K \sqrt{n \log m} \cdot 2n^{1/(4+2p)}} \right\} \\ &\leq Cm^{-K^2/C}, \end{aligned}$$

and hence

$$\max_{1 \leq i \leq m} |\bar{X}_{n,i}| = O_P \left(\sqrt{\frac{\log m}{n}} \right), \quad (16)$$

which together with (14) implies that

$$|M_{n,3} - M_{n,2}| = O_P \left(\frac{\log m}{n} \right) = o_P \left(\sqrt{\frac{1}{n \log m}} \right).$$

Therefore, (15) also holds for $M_{n,3}$.

Step 3: Effect of Estimated Variances Denote by $\check{\sigma}_{n,i,j}$ the estimate of $\tilde{\sigma}_{n,i,j}$

$$\check{\sigma}_{n,i,j} = \frac{1}{n} \sum_{k=1}^n (\tilde{X}_{n,k,i} - \bar{X}_{n,i})(\tilde{X}_{n,k,j} - \bar{X}_{n,j}).$$

In the definition of \tilde{M}_n , $\tilde{\tau}_{n,i,j}$ is unknown, and is estimated by

$$\tilde{\tau}_{n,i,j} = \frac{1}{n} \sum_{k=1}^n \left[(\tilde{X}_{n,k,i} - \bar{X}_{n,i})(\tilde{X}_{n,k,j} - \bar{X}_{n,j}) - \check{\sigma}_{n,i,j} \right]^2$$

In this step we show that (15) holds for \tilde{M}_n . Since

$$n \left| M_{n,3}^2 - \tilde{M}_n^2 \right| \leq n M_{n,3}^2 \cdot \max_{1 \leq i < j \leq m} |1 - \tilde{\tau}_{n,i,j} / \check{\tau}_{n,i,j}|,$$

it suffices to show that

$$\max_{1 \leq i < j \leq m} |\check{\tau}_{n,i,j} - \tilde{\tau}_{n,i,j}| = o_P(1/\log m). \quad (17)$$

Set

$$\begin{aligned} \check{\tau}_{n,i,j,1} &= \frac{1}{n} \sum_{k=1}^n \left[(\tilde{X}_{n,k,i} - \bar{X}_{n,i})(\tilde{X}_{n,k,j} - \bar{X}_{n,j}) - \check{\sigma}_{n,i,j} \right]^2 \\ \check{\tau}_{n,i,j,2} &= \frac{1}{n} \sum_{k=1}^n \left(\tilde{X}_{n,k,i} \tilde{X}_{n,k,j} - \check{\sigma}_{n,i,j} \right)^2. \end{aligned}$$

Observe that

$$\check{\tau}_{n,i,j,1} - \tilde{\tau}_{n,i,j} = (\check{\sigma}_{n,i,j} - \tilde{\sigma}_{n,i,j})^2$$

which in together with (15) implies that

$$\max_{1 \leq i < j \leq m} |\check{\tau}_{n,i,j,1} - \tilde{\tau}_{n,i,j}| = O_P(\log m/n). \quad (18)$$

Note that $\tilde{X}_{n,k,i,j}$ are uniformly bounded according to the truncation (12), so

$$\left(\tilde{X}_{n,k,i} \tilde{X}_{n,k,j} - \tilde{\sigma}_{n,i,j} \right)^2 \leq 64n^{4/(4+2p)}.$$

By Bernstein's inequality, we have

$$\begin{aligned} \max_{1 \leq i < j \leq m} P(|\check{\tau}_{n,i,j,2} - \tilde{\tau}_{n,i,j}| \geq 2n^{-q}) &\leq \exp \left\{ -\frac{2n^{2(1-q)}}{Cn + 2n^{1-q} \cdot 128n^{4/(4+2p)}/3} \right\} \\ &\leq \exp(-n^q/100), \end{aligned}$$

and it follows that

$$\max_{1 \leq i < j \leq m} |\check{\tau}_{n,i,j,2} - \tilde{\tau}_{n,i,j}| = O_P(n^{-q}). \quad (19)$$

In view of (18), (19), and the assumption $\log m = o(n^q)$, we know to show (17), it remains to prove

$$\max_{1 \leq i < j \leq m} |\check{\tau}_{n,i,j,1} - \tilde{\tau}_{n,i,j,2}| = o_P(1/\log m). \quad (20)$$

Elementary calculations show that

$$\max_{1 \leq i < j \leq m} |\check{\tau}_{n,i,j,1} - \tilde{\tau}_{n,i,j,2}| \leq 4h_{n,1}^2 h_{n,2} + 3h_{n,1}^4 + 4h_{n,4}^{1/2} h_{n,2}^{1/2} h_{n,1} + 2h_{n,3} h_{n,1}^2,$$

where

$$\begin{aligned} h_{n,1} &= \max_{1 \leq i \leq m} |\bar{X}_{n,i}| \\ h_{n,2} &= \max_{1 \leq i \leq m} \frac{1}{n} \sum_{k=1}^n \tilde{X}_{n,k,i}^2 \\ h_{n,3} &= \max_{1 \leq i \leq j \leq m} \left| \frac{1}{n} \sum_{k=1}^n \tilde{X}_{n,k,i} \tilde{X}_{n,k,j} - \tilde{\sigma}_{n,i,j} \right| \\ h_{n,4} &= \tilde{\tau}_{n,i,j,2}. \end{aligned}$$

By (16), we know $h_{n,1} = O_P(\sqrt{\log m/n})$. By (19) we have $h_{n,4} = O_P(1)$. Combining (12) and the Bernstein's inequality, we can show that

$$h_{n,3} = O_P\left(\sqrt{\log m/n}\right).$$

As an immediate consequence, we know $h_{n,2} = O_P(1)$. Therefore,

$$\max_{1 \leq i < j \leq m} |\check{\tau}_{n,i,j,1} - \check{\tau}_{n,i,j,2}| = O_P\left(\sqrt{\log m/n}\right),$$

and (20) holds by using the assumption $\log m = o(n^q) = o(n^{1/3})$. The proof of Theorem 2 under (A4') is now complete.

5.3. Proof under (A4)

We follow the proof in Section 5.2, and point out necessary modifications to make it work under (A4). If not specified, all the notations have the same definitions as in Section 5.2. For notational simplicity, we let $p = 4(1 + q) + \delta$.

Step 1: Truncation We truncate $X_{n,k,i}$ by

$$\tilde{X}_{n,k,i} = X_{n,k,i} I \left\{ |X_{n,k,i}| \leq n^{1/4}/\log n \right\},$$

then

$$P\left(\tilde{M}_n \neq M_n\right) \leq nm\mathcal{M}_n(p)n^{-p/4}(\log n)^p \leq C\mathcal{M}_n(p)n^{-\delta/4}(\log n)^p = o(1).$$

Therefore, in the rest of the proof, it suffices to consider $\tilde{X}_{n,k,i}$. For notational simplicity, we still use $\tilde{X}_{n,k,i}$ to denote its centered version with mean zero.

Elementary calculations show that

$$\max_{1 \leq i \leq j \leq m} |\tilde{\sigma}_{n,i,j} - \sigma_{n,i,j}| \leq Cn^{-(p-2)/4}(\log n)^{p-2}, \quad \text{and} \quad (21)$$

$$\max_{\alpha, \beta \in \mathcal{I}_n} \left| \text{Cov}(\tilde{X}_{n,\alpha}, \tilde{X}_{n,\beta}) - \text{Cov}(X_{n,\alpha}, X_{n,\beta}) \right| \leq Cn^{-(p-4)/4}(\log n)^{p-4}. \quad (22)$$

By (21), we know the covariance matrix of $(\tilde{X}_{n,\alpha})_{\alpha \in \mathcal{I}_n}$ satisfies either (B1) or (B2) if Σ_n satisfies (B1) or (B2) correspondingly. Since

$$\mathbb{E}_0 \tilde{X}_{n,\alpha} \in \mathcal{B}\left[1, 8\sqrt{n}/(\log n)^2\right],$$

we know all the conditions of Lemma 6 are satisfied, and hence (15) holds for $M_{n,1}$. Combining (21) and (22), we know (15) also holds with if we replace $M_{n,1}$ by $M_{n,2}$.

Step 2: Effect of Estimated Means Using Bernstein's inequality, we can show

$$\max_{1 \leq i \leq m} |\bar{X}_{n,i}| = O_P\left(\sqrt{\frac{\log n}{n}}\right),$$

which implies that

$$|M_{n,3} - M_{n,2}| = O_P\left(\frac{\log n}{n}\right)$$

and hence (15) also holds for $M_{n,3}$.

Step 3: Effect of Estimated Variances It suffices to show that

$$\max_{1 \leq i < j \leq m} |\check{\tau}_{n,i,j} - \tilde{\tau}_{n,i,j}| = o_P(1/\log n). \quad (23)$$

Using (15), we know

$$\max_{1 \leq i < j \leq m} |\check{\tau}_{n,i,j,1} - \tilde{\tau}_{n,i,j}| = O_P(\log n/n). \quad (24)$$

Since

$$\left(\tilde{X}_{n,k,i} \tilde{X}_{n,k,j} - \tilde{\sigma}_{n,i,j} \right)^2 \leq 64n/(\log n)^4.$$

By Corollary 1.6 of Nagaev (1979) (with $x = n/(\log n)^2$ and $y = n/[2(\log n)^3]$ in their inequality (1.22)), we have

$$\begin{aligned} \max_{1 \leq i < j \leq m} P(|\check{\tau}_{n,i,j,2} - \tilde{\tau}_{n,i,j}| \geq (\log n)^{-2}) &\leq \left[\frac{Cn}{n(\log n)^{-2} \cdot [n(\log n)^{-3}/2]^{q \wedge 1}} \right]^{\log n} \\ &\leq \left[\frac{C(\log n)^5}{n^{q \wedge 1}} \right]^{\log n}, \end{aligned}$$

and it follows that

$$\max_{1 \leq i < j \leq m} |\check{\tau}_{n,i,j,2} - \tilde{\tau}_{n,i,j}| = O_P[(\log n)^{-2}]. \quad (25)$$

In view of (24), (25), we know to show (23), it remains to prove

$$\max_{1 \leq i < j \leq m} |\check{\tau}_{n,i,j,1} - \tilde{\tau}_{n,i,j,2}| = o_P(1/\log n). \quad (26)$$

We know $h_{n,1} = O_P(\sqrt{\log n/n})$ and $h_{n,4} = O_P(1)$. Using the Bernstein's inequality, we can show that

$$h_{n,3} = O_P(\sqrt{\log n/n}),$$

and it follows that $h_{n,2} = O_P(1)$. Therefore,

$$\max_{1 \leq i < j \leq m} |\check{\tau}_{n,i,j,1} - \tilde{\tau}_{n,i,j,2}| = O_P(\sqrt{\log n/n}),$$

and (26) holds. The proof of Theorem 2 under (A4) is now complete.

6. Some auxiliary results

In this section we provide a normal comparison principle and a Gaussian approximation result, and a Poisson convergence theorem.

6.1. A normal comparison principle

Suppose for each $n \geq 1$, $(X_{n,i})_{i \in \mathcal{I}_n}$ is a Gaussian random vector whose entries have mean zero and variance one, where \mathcal{I}_n is an index set with cardinality $|\mathcal{I}_n| = s_n$. Let $\Sigma_n = (r_{n,i,j})_{i,j \in \mathcal{I}_n}$ be the covariance matrix of $(X_{n,i})_{i \in \mathcal{I}_n}$. Assume that $s_n \rightarrow \infty$ as $n \rightarrow \infty$.

We impose either of the following two conditions.

- (B1) For any sequence (b_n) such that $b_n \rightarrow \infty$, $\gamma(n, b_n) = o(1/\log b_n)$;
and $\limsup_{n \rightarrow \infty} \gamma_n < 1$.
- (B2) For any sequence (b_n) such that $b_n \rightarrow \infty$, $\gamma(n, b_n) = o(1)$;
 $\sum_{i \neq j \in \mathcal{I}_n} r_{n,i,j}^2 = O(s_n^{2-\delta})$ for some $\delta > 0$; and $\limsup_{n \rightarrow \infty} \gamma_n < 1$.

where

$$\gamma(n, b_n) := \sup_{i \in \mathcal{I}_n} \sup_{\mathcal{A} \subset \mathcal{I}_n, |\mathcal{A}|=b_n} \inf_{j \in \mathcal{A}} |r_{n,i,j}|$$

$$\text{and } \gamma_n := \sup_{i,j \in \mathcal{I}_n; i \neq j} |r_{n,i,j}|.$$

Lemma 7. Assume either (B1) or (B2). For a positive real number z_n , define

$$A'_{n,i} = \{|X_{n,i}| > z_n\} \quad \text{and} \quad Q'_{n,d} = \sum_{\mathcal{A} \subset \mathcal{I}_n, |\mathcal{A}|=d} P\left(\bigcap_{i \in \mathcal{A}} A'_{n,i}\right).$$

If z_n satisfies that $z_n^2 = 2 \log s_n - \log \log s_n - \log \pi + 2z + o(1)$, then for all $d \geq 1$.

$$\lim_{n \rightarrow \infty} Q'_{n,d} = \frac{e^{-dz}}{d!},$$

Lemma 7 is a refined version of Lemma 20 in Xiao and Wu (2011), so we omit the proof and put the details in a supplementary file.

Remark 2. The conditions imposed on $\gamma(n, b_n)$ seem a little involved. We have the following equivalent versions. Define

$$G_n(t) = \max_{i \in \mathcal{I}_n} \sum_{j \in \mathcal{I}_n} I\{|r_{n,i,j}| > t\}.$$

Then (i) $\gamma(n, b_n) = o(1)$ for any sequence $b_n \rightarrow \infty$ if and only if the sequence $[G_n(t)]_{n \geq 1}$ is bounded for all $t > 0$; and (ii) $\gamma(n, b_n)(\log b_n) = o(1)$ for any sequence $b_n \rightarrow \infty$ if and only if $G_n(t_n) = \exp\{o(1/t_n)\}$ for any positive sequence (t_n) converging to zero.

6.2. A Gaussian approximation result

For a positive integer d , let \mathfrak{B}_d be the Borel σ -field on the Euclidean space \mathbb{R}^d . For two probability measures P and Q on $(\mathbb{R}^d, \mathfrak{B}_d)$ and $\lambda > 0$, define the quantity

$$\pi(P, Q; \lambda) = \sup_{A \in \mathfrak{B}_d} \left\{ \max [P(A) - Q(A^\lambda), Q(A) - P(A^\lambda)] \right\},$$

where A^λ is the λ -neighborhood of A

$$A^\lambda := \left\{ x \in \mathbb{R}^d : \inf_{y \in A} |x - y| < \lambda \right\}.$$

For $\tau > 0$, let $\mathcal{B}(d, \tau)$ be the collection of d -dimensional random variables which satisfy the multivariate analogue of the Bernstein's condition. Denote by (x, y) the inner product of two vectors x and y .

$\mathcal{B}(d, \tau) = \{ \xi \text{ is a random variable} : \mathbb{E}\xi = 0, \text{ and}$

$$|\mathbb{E}[(\xi, t)^2(\xi, u)^{m-2}]| \leq \frac{1}{2} m! \tau^{m-2} \|u\|^{m-2} \mathbb{E}[(\xi, t)^2] \quad (27)$$

for every $m = 3, 4, \dots$ and for all $t, u \in \mathbb{R}^d \}$.

The following Lemma on the Gaussian approximation is taken from Zaitsev (1987).

Lemma 8. *Let $\tau > 0$, and $\xi_1, \xi_2, \dots, \xi_n \in \mathbb{R}^d$ be independent random vectors such that $\xi_i \in \mathcal{B}(d, \tau)$ for $i = 1, 2, \dots, n$. Let $S = \xi_1 + \xi_2 + \dots + \xi_n$, and $\mathcal{L}(S)$ be the induced distribution on \mathbb{R}^d . Let Φ be the Gaussian distribution with the zero mean and the same covariance matrix as that of S . Then for all $\lambda > 0$*

$$\pi[\mathcal{L}(S), \Phi; \lambda] \leq c_{1,d} \exp\left(-\frac{\lambda}{c_{2,d}\tau}\right),$$

where the constants $c_{j,d}$, $j = 1, 2$ may be taken in the form $c_{j,d} = c_j d^{5/2}$.

6.3. Poisson approximation: moment method

Lemma 9. *Suppose for each $n \geq 1$, $(A_{n,i})_{i \in \mathcal{I}_n}$ is a finite collection of events. Let $I_{A_{n,i}}$ be the indicator function of $A_{n,i}$, and $W_n = \sum_{i \in \mathcal{I}} I_{A_{n,i}}$. For each $d \geq 1$, define*

$$Q_{n,d} = \sum_{\mathcal{A} \subset \mathcal{I}_n, |\mathcal{A}|=d} P\left(\bigcap_{i \in \mathcal{A}} A_{n,i}\right).$$

Suppose there exists a $\lambda > 0$ such that

$$\lim_{n \rightarrow \infty} Q_{n,d} = \lambda^d / d! \text{ for each } d \geq 1.$$

Then

$$\lim_{n \rightarrow \infty} P(W_n = k) = \lambda^k e^{-\lambda} / k! \text{ for each } k \geq 0.$$

Observe that for each $d \geq 1$, the d -th factorial moment of W_n is given by

$$\mathbb{E}[W_n(W_n - 1) \cdots (W_n - d + 1)] = d! \cdot Q_{n,d},$$

so Lemma 9 is essentially the moment method. The proof is elementary, and we omit details.

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Supplementary file of Simultaneous Inference of Covariances

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In this document we give the proofs of Lemma 3, Lemma 4 and Lemma 7 of the main article.

Proof of Lemma 3. Assume X_i has mean zero and variance one. Let $\gamma_k = \mathbb{E}(X_0 X_k)$ be the autocovariance of lag k . Then by Proposition 8, Eq. (34) of Xiao and Wu (2011), we know

$$|\gamma_k| \leq \Psi_2 \cdot \Psi_2(|k|). \quad (\text{S.1})$$

- (i) Since $\Psi_4 < \infty$, we know for any $\eta > 0$, there exists a $N_1 > 0$ such that $|\gamma_k| < \eta$ when $k \geq N_1$. For $j \leq k$, define $\tilde{X}_{k,j} = g(\epsilon_k, \dots, \epsilon_{j+1}, \epsilon'_j, \epsilon'_{j-1}, \dots)$, where $(\epsilon'_i)_{i \in \mathbb{Z}}$ is an i.i.d. copy of $(\epsilon_i)_{i \in \mathbb{Z}}$. By Eq. (38) of Xiao and Wu (2011), we know there exists a $N_2 > 0$ such that when $k \geq N_2$, $\|X_k - \tilde{X}_k\|_4 \leq \eta$. Set $N = \max\{N_1, N_2\}$, when $k \geq N$, we have

$$\begin{aligned} \text{Var}(X_0 X_k) &= \mathbb{E}(X_0^2 X_k^2) - \gamma_k^2 = \mathbb{E}(X_k^2 X_{k,j}^2) + \mathbb{E}[X_0^2 (X_k^2 - X_{k,j}^2)] - \gamma_k^2 \\ &\geq 1 - \eta^2 - 2\|X_0\|_4^3 \cdot \eta. \end{aligned}$$

Therefore, (A1) holds because η can be arbitrarily small.

- (ii) We need to show that

$$\sup_{j \geq 0, 0 \leq k \leq l, (0,j) \neq (k,l)} \text{Cor}(X_0 X_j, X_k X_l) < 1.$$

It suffices to show that for some $N > 0$

$$\sup_{j \geq 0, 0 \leq k \leq l, (0,j) \neq (k,l), j+k+l \geq N} \text{Cor}(X_0 X_j, X_k X_l) < 1.$$

If $j+k+l \geq N$, then the set $\{0, j, k, l\}$ can be partitioned into two non-empty subsets \mathcal{B}_1 and \mathcal{B}_2 whose distance is no less than $N/6$. We only consider this type of partitions. If there is a partition such that

one of \mathcal{B}_1 and \mathcal{B}_2 has cardinality one, then similarly as (i), we know for any $\eta > 0$, when N is large enough,

$$|\text{Cov}(X_0X_j, X_kX_l)| = |\mathbb{E}(X_0X_jX_kX_l) - \gamma_j\gamma_{l-k}| \leq \eta.$$

If for any partition both \mathcal{B}_1 and \mathcal{B}_2 has cardinality two, there are two sub-cases. (a) $j < k \leq l$ and $k - j \geq N/6$. For any $\eta > 0$, when N is large enough, we have

$$|\text{Cov}(X_0X_j, X_kX_l)| = |\mathbb{E}[X_0X_j(X_kX_l - X_{k,j}X_{l,j})]| \leq \eta.$$

(b) $\min\{j, l\} - k \geq N/6$. As in (i), for any $\eta > 0$, when N is large enough, we have $\text{Var}(X_0X_j) \geq 1 - \eta$, $\text{Var}(X_kX_l) \geq 1 - \eta$, and $|\gamma_j\gamma_{l-k}| < \eta$. On the other hand, the condition $\Psi_4 > 0$ guarantees that the process is non-deterministic, and hence $\gamma := \sup_{t \geq 1} |\gamma_t| < 1$. It follows that when N is large enough

$$\begin{aligned} |\mathbb{E}(X_0X_jX_kX_l)| &= |\mathbb{E}(X_0X_{j,k}X_kX_{l,k}) + \mathbb{E}[X_0X_k(X_jX_l - X_{j,k}X_{l,k})]| \\ &\leq \gamma + \eta. \end{aligned}$$

Therefore,

$$|\text{Cor}(X_0X_j, X_kX_l)| \leq (\gamma + 2\eta)/(1 - \eta) < 1$$

when η is small enough. The proof of (ii) is now complete.

(iii) We first consider (A3). Note that

$$\text{Cov}(X_iX_j, X_kX_l) = \text{Cum}(X_i, X_j, X_k, X_l) + \gamma_{i-k}\gamma_{j-l} + \gamma_{i-l}\gamma_{j-k},$$

where $\text{Cum}(X_i, X_j, X_k, X_l)$ is the fourth order joint cumulant of the random vector $(X_i, X_j, X_k, X_l)^\top$. Fix a subset $\{i, j\}$, for any integer $b > 0$, there are at most $8b^2$ subsets $\{k, l\}$ such that $\{k, l\} \subset B(i; b) \cup B(j; b)$, where $B(x; r)$ is the open ball $\{y : |x - y| < r\}$. For all other subsets $\{k, l\}$, by (S.1), we have

$$|\gamma_{i-k}\gamma_{j-l} + \gamma_{i-l}\gamma_{j-k}| \leq C\Psi_4(b).$$

On the other hand, using similar arguments as Theorem 21 of Xiao and Wu (2011), we can show that

$$|\text{Cum}(X_i, X_j, X_k, X_l)| \leq C\Psi_4(\lfloor b/2 \rfloor).$$

Therefore, if $\Psi_4(k) = o(1/\log k)$ as $k \rightarrow \infty$, then (A3) holds.

Now we turn to (A3'). Write

$$\text{Cov}(X_iX_j, X_kX_l) = \mathbb{E}(X_iX_jX_kX_l) - \gamma_{i-j}\gamma_{k-l}.$$

By (S.1), it is easily seen that

$$\sum_{1 \leq i, j, k, l \leq m} \gamma_{i-j}^2 \gamma_{k-l}^2 = O(m^{4-2\delta}).$$

It then suffices to show

$$\sum_{1 \leq i \leq j \leq k \leq l \leq m} [\mathbb{E}(X_i X_j X_k X_l)]^2 = O(m^{4-\delta}),$$

which is true because by Eq. (38) of Xiao and Wu (2011)

$$[\mathbb{E}(X_i X_j X_k X_l)]^2 = [\mathbb{E}(X_i X_j X_k (X_l - X_{l,k}))]^2 \leq 12 \|X_0\|_4^6 [\Psi_4(l - k)]^2.$$

The proof of Lemma 3 is now complete. \square

We now give the proof of Lemma 4.

Proof of Lemma 4. Suppose (Y_1, Y_2, Y_3, Y_4) has a joint normal distribution. We can write $Y_i = \alpha_i^\top \mathbf{Z}$, where \mathbf{Z} is a four dimensional standard Gaussian random vector. For any $0 < \nu < 1$, define the subset of \mathbb{R}^{16} ,

$$D_\nu = \{(\alpha_1^\top, \alpha_2^\top, \alpha_3^\top, \alpha_4^\top) : |\alpha_i|^2 = 1 \text{ and } |\alpha_i^\top \alpha_j| \leq 1 - \nu \text{ for } 1 \leq i \neq j \leq 4.\}$$

Since $|\text{Cor}(Y_1 Y_2, Y_3 Y_4)|$ is a continuous function on D_ν , and D_ν is compact, the maximum correlation is attained at some point in D_ν .

On the other hand, elementary calculation shows that $\text{Cor}(Y_1 Y_2, Y_3 Y_4) = 1$ if and only if Y_1, Y_2, Y_3, Y_4 are all perfectly correlated. The proof is now complete. \square

The proof of Lemma 7 is a refined version of that of Lemma 20 in Xiao and Wu (2011). We need the following bounds on normal tail probabilities, which are taken from Lemma 19 of Xiao and Wu (2011).

Denote by $\varphi_d((r_{ij}); x_1, \dots, x_d)$ the density of a d -dimensional multivariate normal random vector $\mathbf{X} = (X_1, \dots, X_d)^\top$ with mean zero and covariance matrix (r_{ij}) , where we always assume $r_{ii} = 1$ for $1 \leq i \leq d$ and (r_{ij}) is nonsingular. Let

$$Q_d((r_{ij}); z) = \int_z^\infty \cdots \int_z^\infty \varphi_d((r_{ij}), x_1, \dots, x_d) dx_d \cdots dx_1.$$

Lemma S.1. *For every $z > 0$, $0 < s < 1$, $d \geq 1$ and $\epsilon > 0$, there exists positive constants C_d and ϵ_d such that for $0 < \epsilon < \epsilon_d$*

1. *if $|r_{ij}| < \epsilon$ for all $1 \leq i < j \leq d$, then*

$$Q_d((r_{ij}); z) \leq C_d f_d(\epsilon, 1/z) \exp \left\{ - \left(\frac{d}{2} - C_d \epsilon \right) z^2 \right\} \quad (\text{S.2})$$

where $f_{2k}(x, y) = \sum_{l=0}^k x^l y^{2(k-l)}$ and $f_{2k-1}(x, y) = \sum_{l=0}^{k-1} x^l y^{2(k-l)-1}$ for $k \geq 1$;

2. *if for all $1 \leq i < j \leq d+1$ such that $(i, j) \neq (1, 2)$, $|r_{ij}| \leq \epsilon$, then*

$$Q_{d+1}((r_{ij}); z) \leq C_d \exp \left\{ - \left(\frac{(1 - |r_{12}|)^2 + d}{2} - C_d \epsilon \right) z^2 \right\}. \quad (\text{S.3})$$

We first give a one-sided version of Lemma 7 and its proof, then we show how it implies Lemma 7.

Lemma S.2. Assume either (B1) or (B2). For a positive real number z_n , define the event $A_{n,i}$ and $Q_{n,d}$ as

$$A_{n,i} = \{X_{n,i} > z_n\} \quad \text{and} \quad Q_{n,d} = \sum_{\mathcal{A} \subset \mathcal{I}_n, |\mathcal{A}|=d} P\left(\bigcap_{i \in \mathcal{A}} A_{n,i}\right).$$

If z_n satisfies that $z_n^2 = 2 \log s_n - \log \log s_n - \log(4\pi) + 2z + o(1)$, then for all $d \geq 1$

$$\lim_{n \rightarrow \infty} Q_{n,d} = \frac{e^{-dz}}{d!}.$$

Proof. The following facts about normal tail probabilities are well-known:

$$P(X_1 \geq x) \leq \frac{1}{\sqrt{2\pi}x} e^{-x^2/2} \text{ for } x > 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{P(X_1 \geq x)}{(1/x)(2\pi)^{-1/2} \exp\{-x^2/2\}} = 1, \quad (\text{S.4})$$

By the assumption on z_n , if for each n , $X_{n,i}$, $i \in \mathcal{I}_n$ are i.i.d., then by (S.4),

$$\begin{aligned} \lim_{n \rightarrow \infty} Q_{n,d} &= \lim_{n \rightarrow \infty} \binom{n}{d} Q_d(I_d, z_n) \\ &= \lim_{n \rightarrow \infty} \binom{n}{d} \frac{1}{(2\pi)^{d/2} z_n^d} \exp\left\{-\frac{dz_n^2}{2}\right\} = \frac{e^{-dz}}{d!}. \end{aligned}$$

When the $X_{n,i}$'s are dependent, the result is still trivially true when $d = 1$. Now we deal with the $d \geq 2$ case. Suppose (b_n) is a sequence of positive numbers which converges to infinity. For each subset J of \mathcal{I}_n with cardinality $|J| = d$, we define an undirected graph $\mathcal{G}(J)$ by identifying each $i \in J$ with a node and saying i and j are adjacent if $|r_{n,i,j}| > \gamma(n, b_n)$. Suppose the graph $\mathcal{G}(J)$ has $d - s$ connected components $\mathcal{B}_1, \dots, \mathcal{B}_{d-s}$. If $s \geq 1$, assume w.l.o.g. that $|\mathcal{B}_1| \geq 2$. Pick $k_0, k_1 \in \mathcal{B}_1$, and $k_p \in \mathcal{B}_p$ for $2 \leq p \leq d - s$, and set $K = \{k_0, k_1, k_2, \dots, k_{d-s}\}$. Define $Q_J = P(\cap_{k \in J} A_k)$ and Q_K similarly, then $Q_J \leq Q_K$. By (S.3) of Lemma S.1, there exists a number $M > 1$ depending on d and the sequences (γ_n) and (b_n) , such that when $n \geq M$,

$$\begin{aligned} Q_K &\leq C_{d-s} \exp\left\{-\left(\frac{(1-\gamma_n)^2 + d-s}{2} - C_{d-s}\gamma(n, b_n)\right) z_n^2\right\} \\ &\leq C_{d-s} \exp\left\{-\left(\frac{d-s}{2} + \frac{(1-\gamma_n)^2}{3}\right) z_n^2\right\}. \end{aligned}$$

Note that $z_n^2 = 2 \log s_n - \log \log s_n + O(1)$. Pick $b_n = \lfloor s_n^\alpha \rfloor$ for some $\alpha < (1 - \gamma_n)^2 / 3d$. For any $1 \leq a \leq d - 1$, since there are at most $O(b_n^a s_n^{d-a})$ subsets $J \subset \mathcal{I}_n$ such that $|J| = d$ and the graph $\mathcal{G}(J)$ has $d - a$ connected components, we know the sum of Q_J over these J is dominated by

$$C_{d-a} \exp\left\{\log s_n \left((d-a) + \frac{2(d-1)(1-\gamma_n)^2}{3d} - (d-a) - \frac{2(1-\gamma_n)^2}{3}\right)\right\}$$

when n is large enough, which converges to zero. Therefore, it remains to consider all the subsets $J \subset \mathcal{I}_n$ such that the graph $\mathcal{G}(J)$ has no edges

Let $J \subset \mathcal{I}_n$ be a subset such that $|J| = d$, and $|r_{n,i,j}| < \gamma(n, b_n)$ for all pairs i, j such that $i, j \in J$ and $i \neq j$, and $\mathcal{J}(d, b_n)$ be the collection of all such subsets. Let $(r_{ij})_{i,j \in J}$ be the d -dimensional covariance matrix of $\mathbf{X}_J := (X_{n,i})_{i \in J}$. There exists a matrix $R_J = \theta(r_{ij})_{i,j \in J} + (1 - \theta)I_d$ for some $0 < \theta < 1$ such that

$$Q_J - Q_d(I_d, z_n) = \sum_{h,l \in J, h < l} \frac{\partial Q_d}{\partial r_{hl}}[R_J; z_n] r_{hl}.$$

4

Let R_H , $H = J \setminus \{h, l\}$, be the correlation matrix of the conditional distribution of \mathbf{X}_H given X_h and X_l .

By (S.2) of Lemma S.1, for n large enough

$$\begin{aligned}
\frac{\partial Q_d}{\partial r_{hl}}[R_J; z_n] &\leq C \exp \left\{ -\frac{z_n^2}{1 + |r_{n,h,l}|} \right\} \cdot Q_{d-2}(R_K; (1 - 3\gamma(n, b_n))z_n) \\
&\leq CC_{d-2}f_{d-2}(\gamma(n, b_n), 1/z_n) \exp \left\{ -\frac{z_n^2}{1 + |r_{n,h,l}|} \right\} \\
&\quad \times \exp \left\{ -\left(\frac{d-2}{2} - 2C_{d-2}\gamma(n, b_n) \right) (1 - 3\gamma(n, b_n))^2 z_n^2 \right\} \\
&\leq C_d f_{d-2}(\gamma(n, b_n), 1/z_n) \\
&\quad \times \exp \left\{ -\left(\frac{d}{2} - (2C_{d-2} + 3(d-2))\gamma(n, b_n) - |r_{n,h,l}| \right) z_n^2 \right\} \\
&\leq C_d f_{d-2}(\gamma(n, b_n), 1/z_n) \exp \left\{ -\left(\frac{d}{2} - C_d \gamma(n, b_n) \right) z_n^2 \right\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{J \in \mathcal{J}(d, b_n)} |Q_J - Q_d(I_d; z_n)| \\
&\leq C_d f_{d-2}(\gamma(n, b_n), 1/z_n) \\
&\quad \times \sum_{J \in \mathcal{J}(d, b_n)} \sum_{i,j \in J; i \neq j} \exp \left\{ -\left(\frac{d}{2} - C_d \gamma(n, b_n) \right) z_n^2 \right\} |r_{n,i,j}| \\
&\leq C_d f_{d-2}(\gamma(n, b_n), 1/z_n) s_n^{d-2} \\
&\quad \times \sum_{i,j \in \mathcal{I}_n}^* \exp \left\{ -\left(\frac{d}{2} - C_d \gamma(n, b_n) \right) z_n^2 \right\} |r_{n,i,j}|,
\end{aligned} \tag{S.5}$$

where the sum $\sum_{i,j \in \mathcal{I}_n}^*$ is over all the pair (i, j) such that $|r_{n,i,j}| \leq \gamma(n, b_n)$. Under the assumption (B1), we have

$$\begin{aligned}
&\sum_{J \in \mathcal{J}(d, b_n)} |Q_J - Q_d(I_d; z_n)| \\
&\leq C_d f_{d-2}(\gamma(n, b_n), 1/z_n) (\log s_n)^{d/2} \gamma(n, b_n) \exp \{C_d \gamma(n, b_n) (\log s_n)\}
\end{aligned} \tag{S.6}$$

Since $\lim_{n \rightarrow \infty} \gamma(n, b_n) \log b_n = 0$, it also holds that $\lim_{n \rightarrow \infty} \gamma(n, b_n) \log s_n = 0$. Note that $\lim_{n \rightarrow \infty} (\log s_n)^{1/2}/z_n = 2^{-1/2}$, it follows that $\lim_{n \rightarrow \infty} f_{d-2}(\gamma(n, b_n), 1/z_n) (\log s_n)^{d/2-1} = 2^{-d/2+1}$. Therefore, the term in (S.6) converges to zero, and the theorem holds under (B1).

Alternatively, if (B2) is true, from (S.5) we have

$$\begin{aligned}
&\sum_{J \in \mathcal{J}(d, b_n)} |Q_J - Q_d(I_d; z_n)| \\
&\leq C_d f_{d-2}(\gamma(n, b_n), 1/z_n) s_n^{-2} (\log s_n)^{d/2} \sum_{i,j \in \mathcal{I}_n}^* \exp \{C_d \gamma(n, b_n) (\log s_n)\} |r_{n,i,j}| \\
&\leq C_d f_{d-2}(\gamma(n, b_n), 1/z_n) s_n^{-1} (\log s_n)^{d/2} \exp \{C_d \gamma(n, b_n) (\log s_n)\} \left(\sum_{i,j \in \mathcal{I}_n} r_{n,i,j}^2 \right)^{1/2} \\
&\leq C_d s_n^{-\delta/2} (\log s_n) \exp \{C_d \gamma(n, b_n) (\log s_n)\} = o(1),
\end{aligned}$$

and the proof is complete. □

Now we give the proof of Lemma 7.

Proof of Lemma 7. In the proof of Theorem S.2, the upper bounds on Q_J and $|Q_J - Q(I_d; z_n)|$ are expressed through the absolute values of the covariances, so we can obtain the same bounds for probabilities of the form $P(\cap_{1 \leq i \leq d} \{(-1)^{a_i} X_{t_i} \geq z_n\})$ for any $(a_1, \dots, a_d) \in \{0, 1\}^d$. Based on this observation, Lemma 7 is an immediate consequence of Lemma S.2. □

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Han Xiao and Wei Biao Wu. Asymptotic inference of autocovariances of stationary processes. *preprint*, available at <http://arxiv.org/abs/1105.3423>, 2011.